IFTST 1 Logic

1.1 Calculate, according to truth - tables:
a) $(\mathbf{0} \Rightarrow \mathbf{1}) \Rightarrow [\mathbf{0} \wedge \sim (\mathbf{0} \vee \mathbf{1})]$,
b) $(\mathbf{0} \Rightarrow \mathbf{0}) \Rightarrow [\mathbf{0} \Rightarrow (\mathbf{0} \Rightarrow \mathbf{0})]$,
c) $[(\mathbf{1} \Rightarrow \mathbf{0}) \Rightarrow \mathbf{1}] \Rightarrow (\mathbf{1} \Rightarrow \mathbf{0})$,
d) $[(\mathbf{0} \vee \mathbf{1}) \wedge \mathbf{0}] \vee \mathbf{0}$,
e) $(\mathbf{1} \Rightarrow \mathbf{0}) \Leftrightarrow [(\mathbf{0} \Rightarrow \mathbf{1}) \vee \mathbf{0}]$.

1.2 Let $x, y, z, t \in \{\mathbf{0}, \mathbf{1}\}$. Solve the following equations:
a) $(x \wedge y) \Rightarrow (z \vee t) = \mathbf{0}$
b) $\sim (x \Rightarrow y) \wedge (x \vee y) = \mathbf{1}$
c) $\sim [(x \vee y \vee t) \Rightarrow (\sim x \vee z \vee \sim t)] = \mathbf{0}$
d) $(x \Leftrightarrow y) \wedge (y \Leftrightarrow \sim z) \wedge (z \Leftrightarrow \sim t) = \mathbf{1}$,
e) $(x \Rightarrow y) \vee (y \Rightarrow z) \vee (z \Rightarrow t) \vee (t \Rightarrow x) = \mathbf{0}$.

1.3 For how many assignments the formula is true? Transform it into a) DNF form (e.i. $(x_1 \wedge x_2 \wedge x_3) \vee (..)... \vee (...)$ where x_i are variable or their negations) b) CNF form (e.i. $(x_1 \vee x_2 \vee x_3) \wedge (..)... \wedge (...)$:
a) $[p \vee (q \Rightarrow r)] \wedge [\sim r \vee (q \Rightarrow p)]$
b) $\sim [p \wedge (r \Rightarrow q)] \vee [q \Rightarrow (p \wedge r)]$
c) $p \Rightarrow [\sim p \Rightarrow (p \Rightarrow \sim p)]$
d) $[(p \Rightarrow \sim p) \Rightarrow p] \Rightarrow \sim p$
e) $(p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee q) \wedge (\sim p \vee \sim q)$
f) $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$
g) $[(p \Rightarrow q) \Rightarrow r] \Rightarrow [(p \Rightarrow r) \Rightarrow (q \Rightarrow r)]$
h) $[(p \Rightarrow q) \Rightarrow (r \Rightarrow q)] \Rightarrow [(p \Rightarrow r) \Rightarrow q)]$
i) $[(p \Rightarrow q) \Rightarrow \sim (p \Rightarrow r)] \Rightarrow [\sim (p \Rightarrow r) \Rightarrow (p \Rightarrow q)]$
j) $[(p \Rightarrow \sim q) \Rightarrow \sim p] \Rightarrow [p \Rightarrow (q \Rightarrow p)]$
k) $\sim [(p \vee q \vee r) \Rightarrow (\sim p \wedge \sim q \wedge \sim r)]$
l) $(p \wedge q \wedge \sim r) \Leftrightarrow [(\sim p \Rightarrow \sim q) \vee r]$
m) $(p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim r \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r)$
n) $[(p \Rightarrow q) \Rightarrow r] \vee [(\sim r \vee q) \wedge r]$,
o) $(p \Rightarrow q) \Rightarrow [(q \Rightarrow r) \Rightarrow (p \Rightarrow r)]$.

1.4 Define the connectives: negation, disjunction, conjunction and implication in terms of
a) Sheffer's functor $|$, where $p \mid q \Leftrightarrow \sim p \vee \sim q$
b) Peirce's functor $\downarrow$, where $p \downarrow q \Leftrightarrow \sim p \wedge \sim q$.

1.5 Write the mathematical formulas corresponding to the following statements with the help of the following signs only: propositional connectives, quantifiers, variables varied through set $\mathbb{N}$ and symbols indicated in brackets (define auxiliary symbols if necessary)
a) *x is a divisor of y* (symbols: $=, <, \cdot$),
b) *x is even* $(=, +)$,
c) *x is odd* $(=, +)$,
d) *x is divisible by 3* $(=, +)$,
e) *x is product of two odd numbers* $(=, \cdot, +)$,
f) *x has odd divisors only* $(=, \cdot, +)$,
g) *x is a prime number* $(=, <, \cdot, 1)$,
h) *any two numbers have the least common multiple* $(=, <, \cdot)$,
i) *there doesn't exist the greatest number* $(\leq)$,

j) *every even number is a sum of two squares,* $(=,\cdot,+)$,
k) *there exists a number with three divisors only* $(=,\cdot)$,
l) *there exists one and only one prime number between 10 and 20* $(=,<,\cdot,1)$,
m) *there are no numbers with at most two multiples being squares* $(=,<,\cdot)$,
n) *there exists a number which is not a square of an odd number* $(=,\cdot,+)$,
o) *for any odd number there exists a greater even number* $(<,+)$,
p) *the product of prime numbers is a sum of three prime numbers* $(=,<,\cdot,+,1)$,
r) x *is greater than* y $(=,+,\cdot)$

1.6 The same claim with variables varied through set $\mathbb{R}$:
a) *there are no negative squares* $(<,\cdot,0)$,
b) *the product of two numbers with different signs is a positive number* $(<,\cdot,0)$,
c) *each positive number has a square root* $(=,<,\cdot,0)$,
d) *each linear equation has a solution* $(=,0,\cdot,+)$,
e) *each linear equation has a unique solution* $(=,0,\cdot,+)$,
f) *there exist a quadratic polynomial with exactly two solutions* $(=,0,\cdot,+)$,
g) *each system of two linear equations with two variables has a unique solution* $(=,0,\cdot,+)$,
h) *the set of all positive reals is closed with respect to the operation of division* $(=,>,\cdot,0)$.
i) *not every real is a value of a quadratic polynomial with positive coefficients* $(=,+,\cdot,0,>)$.
j) x *is negative* $(\cdot,>)$.

1.7 Which of the following formulas are true? For the false formulas show counterexamples.
a) $[(\exists x)\phi(x) \wedge (\exists x)\psi(x)] \Rightarrow (\exists x)(\phi(x) \wedge \psi(x))$,
b) $(\exists x)[\phi(x) \wedge \psi(x)] \Rightarrow [(\exists x)\phi(x) \wedge (\exists x)\psi(x)]$,
c) $[(\forall x)\phi(x) \vee (\forall x)\psi(x)] \Rightarrow (\forall x)(\phi(x) \vee \psi(x))$,
d) $(\forall x)(\phi(x) \vee \psi(x)) \Rightarrow [(\forall x)\phi(x) \wedge (\forall x)\psi(x)]$,
e) $(\forall x)(\exists y)\phi(x,y) \Rightarrow (\exists y)(\forall x)\phi(x,y)$,
f) $(\exists y)(\forall x)\phi(x,y) \Rightarrow (\forall x)(\exists y)\phi(x,y)$,
g) $[(\exists x)(\phi(x) \Rightarrow \psi(x))] \Rightarrow [(\exists x)\phi(x) \Rightarrow (\exists x)\psi(x)]$.

1.8 Reduce the formulas in Exercise 1.8 to the form where all quantifiers are in front of the formula.

1.9 Prove the following rules for formulas with bounded quantifiers (derive them from suitable logical rules for unbounded quatifiers):
a) $[(\forall x)_{\alpha(x)}\phi(x) \wedge (\forall x)_{\alpha(x)}\psi(x)] \Leftrightarrow (\forall x)_{\alpha(x)}(\phi(x) \wedge \psi(x))$
b) $\sim (\exists x)_{\alpha(x)}\phi(x) \Leftrightarrow (\forall x)_{\alpha(x)} \sim \phi(x)$

1.11 Interpret symbols $\alpha, *$ occurring in following formulas to make them (1) true, (2) false.
a) $(\exists x)(\forall y)\alpha(x,y)$
b) $(\forall x,y,z)[\alpha(x,y) \wedge \alpha(y,z) \Rightarrow \alpha(x,z)]$
c) $(\forall x,y)\alpha(x,y) \Leftrightarrow (\forall x)\alpha(x,x)$
d) $(\exists x)(\forall y)x * y = y$